# ON A METHOD OF INVESTIGATING THE STABILITY OF A NULL-SOLUTION IN DOUBTFUL CASES

## (OB ODNOM METODE ISSLEDOVANIIA USTOICHIVOSTI NULEVOGO RESHENIIA V SOMNITEL'NYKH SLUCHAIAKH)

*PMM Vol.22, No.1, 1958, pp. 46-49.*

V.I. ZUBOV
(Leningrad)

*(Received 29 April 1957)*

A procedure of analyzing the stability of a null-solution of a system of $n + k$ ordinary differential equations, applicable in doubtful cases, is considered.

This procedure consists in the study of the stability of the null-solution separately for $k$ and for $n$ equations resulting from the initial system.

Let us consider the system

$$\frac{dy_s}{dt} = f_s(x_1, \ldots, x_n, y_1, \ldots, y_k, t) \qquad (s = 1, \ldots, k)$$
$$\frac{dx_j}{dt} = g_j(x_1, \ldots, x_n, y_1, \ldots, y_k, t) \qquad (j = 1, \ldots, n) \tag{1}$$

We assume that the functions $f_s$ and $g_j$ are given continuous functions in the region $|X| \leqslant H$, $|Y| \leqslant H$, $t \geqslant 0$.

Furthermore we assume

$$f_s \equiv 0 \quad \text{for} \quad |Y| = 0$$
$$g_j \equiv 0 \quad \text{for} \quad |X| = |Y| = 0 \quad (s = 1, \ldots, k;\ j = 1, \ldots, n)$$

*Definition 1.* A null-solution of the system (1) is called stable according to Liapunov if for any $\epsilon > 0$ we can find $\delta(\epsilon) > 0$ such that for $|X^{(0)}| < \delta$, $|Y^{(0)}| < \delta$ we have $|X(t, X^{(0)}, Y^{(0)}, t_0)| < \epsilon$, $|Y(t, X^{(0)}, Y^{(0)}, t_0)| < \epsilon$ for $0 \leqslant t_0 \leqslant t$.

Here $X(t, X^{(0)}, Y^{(0)}, t_0)$, $Y(t, X^{(0)}, Y^{(0)}, t_0)$ indicate the set of functions $x_1, \ldots, x_n, y_1, \ldots, y_k$ representing the solution of the system (1), subjected to the conditions

$$x_i = x_i^{(0)}, \quad y_j = y_j^{(0)} \quad \text{for} \quad t = t_0 \quad (i = 1, \ldots, n;\ j = 1, \ldots, k)$$

If a null-solution of the system (1) is stable and $X(t, X^{(0)}, Y^{(0)}, t_0) \to 0$, $Y(t, X^{(0)}, Y^{(0)}, t_0) \to 0$ as $t \to +\infty$, then such a null-solution is called asymptotically stable. If in the first group of equations of

Reprint Order No. PMM 4.

system (1) the quantities $x_1, \ldots, x_n$ are replaced by the continuously differentiable functions $x_1(t), \ldots, x_n(t)$, which are given for $t \geqslant 0$, such that $|X(t)| < H$, then we obtain a system of $k$ differential equations of the type

$$\frac{dy_s^0}{dt} = f_s(t, x_1(t), \ldots, x_n(t), \quad y_1^0, \ldots, y_k^0) \tag{3}$$

possessing a null-solution.

*Definition* 2. A null-solution of system (3) is called strongly stable, if we can find a number $H_1 > 0$ such that for every $\epsilon_1 > 0$ there exists a $\delta_1 > 0$ characterized by $|Y^0(t, Y^{(0)}, t_0| < \epsilon_1$ for $0 \leqslant t_0 \leqslant t$ and $|Y^{(0)}| < \delta_1$ for any continuously differentiable functions $x_1(t), \ldots, x_n(t)$ given for $t \geqslant 0$ and $|X| < H_1$. If, in addition, $Y^0 \to 0$ as $t \to +\infty$, then the null-solution of system (2) will be called strongly asymptotically stable.

Let us introduce a function

$$W(t, x_1, \ldots, x_n, y_1, \ldots, y_k)$$

*Definition 3.* We shall say that the function $W(t, x_1, \ldots, x_n, y_1, \ldots, y_k)$ is "strictly negative-definite" with respect to $X$ if it is possible to find a function

$$\varphi_s(x_1, \ldots, x_n) > 0 \qquad \text{for } X \neq 0 \; (s = 1, \ldots, k)$$

such that the function $W(t, x_1, \ldots, x_n, y_1(x_1, \ldots, x_n) \ldots, Y_k(x_1, \ldots, x_n))$ will be negative-definite for any choice of continuous functions $y_s(x_1, \ldots, x_n)$ satisfying the condition

$$|y_s(x_1, \ldots, x_n)| < \varphi_s(x_1, \ldots, x_n) \qquad (s = 1, \ldots, k)$$

For example, the function $W = -x^2 + y \sin t$ will be strictly negative-definite. Here it is possible to assume $\phi = \frac{1}{2} x^2$

*Theorem 1.* If:

(1) A null-solution of system (2) is strongly stable (strongly asymptotically stable),

(2) there exists a continuously differentiable positive-definite function $V(t, x_1, \ldots, x_n)$, uniformly continuous with respect to $t$ for $X = 0$, $V(t, X) \to 0$ as $X \to 0$ uniformly for $t \geqslant 0$

(3) the function

$$W(t, x_1, \ldots, x_n, y_1, \ldots, y_k) = \frac{\partial V}{\partial t} + \sum_{j=1}^{n} \frac{\partial V}{\partial x_j} g_j(t, X, Y)$$

is "strictly negative-definite" with respect to $X$, then the null-solution of the system (1) will also be stable (asymptotically stable).

*Proof.* According to condition (2), there exists a number $h > 0$ and $h < H_1$ such that for $\epsilon > 0$

$$\inf V(t, X) = m_1(\varepsilon) > 0 \qquad (t \geqslant 0, \varepsilon \leqslant |X| \leqslant H)$$

Let us take a certain number $\epsilon > 0$, $\epsilon > H$ and choose a positive number $m < m_1(\epsilon)$.

According to condition 2 of Theorem 1 there exists a number $\lambda < \epsilon$ such that the function $V(t,X) < m$ for $|X| < \lambda$, $t \geqslant 0$.

On the strength of condition (3) there exists a number $\epsilon_1 \leqslant \epsilon$ such that for $|Y| < \epsilon_1$ we shall have $W(t,X,Y) < 0$ for $t \geqslant 0$ and $\lambda \leqslant |X| \leqslant \epsilon$.

On the strength of condition (1) for a number $\epsilon_1 > 0$ it is possible to find a number $\delta_1 > 0$, connected with $\epsilon_1$ by the relation indicated in Definition 2.

Let us assume $\delta = \min(\lambda, \delta_1) < \epsilon$. We show that for $|X^{(0)}| < \delta$, $|Y^{(0)}| < \delta$ inequality (2) is fulfilled.

Assume that this is not true. Then it is possible to find a number $T$ such that

$$|X(t, X^{(0)}, Y^{(0)}, t_0)| < \varepsilon, \quad t \in [t_0, T], \quad |X(T, X^{(0)}, Y^{(0)}, t_0)| = \varepsilon$$

There follows from Definition 2:

$$|Y(t, X^{(0)}, Y^{(0)}, t_0)| < \varepsilon_1 < \varepsilon \quad \text{for } t \in [t_0, T]$$

since the set of the functions $Y(t,X^{(0)},Y^{(0)},t_0)$ can be assumed to be the solution of system (3) in the time interval $[t_0,T]$, in which functions $x_j(t,X^{(0)},Y^{(0)},t_0)$ are selected for the function $x_j(t)$.

Let us designate by $V(t)$ the value of the function $V(t,X)$ on the integral line under the study. Clearly $V(t_0) < m$ but $V(T) > m$. Function $V(t)$ is continuously differentiable, therefore there exists a number $t_2$ such that $V(t_1) = m$ and $V(t) > m$ for $t_1 < t \leqslant T$. Then $[dV/dt]_{t=t_1} \geqslant 0$ for $t = t_1$, when inequality $\lambda \leqslant |X(t_1,X^{(0)},Y^{(0)},t_0)| \leqslant \epsilon$ is satisfied and consequently $[dV/dt]_{t=t_1} < 0$. This contradiction proves inequality (2), and then there follows from the Condition (1) and Definition 2,

$$|Y(t, X^{(0)}, Y^{(0)}, t_0)| < \varepsilon_1 \leqslant \varepsilon \quad \text{for } t \geqslant t_0 \geqslant 0$$

Thus, the null-solution of the system (1) is stable.

If the null-solution of system (3) is asymptotically stable, then

$$Y(t, X^{(0)}, Y^{(0)}, t_0) \to 0 \quad \text{as} \quad t \to +\infty$$

Let $|X(t,X^{(0)},Y^{(0)},t_0)| > \alpha > 0$ for $t$ as $t_0$. Then there exists a number $r > t_0$ such that $W(t,X(t,X^0,Y^0,t_0),\ Y(t,X^0,Y^{(0)},t_0)) < -\sigma < 0$ for $t \geqslant r$, therefore $V(t) \leqslant V(r) - \sigma(t - r)$ for $t \geqslant r$, which is impossible.

Hence $X(t,X^{(0)},Y^{(0)},t_0) \to 0$ for $t \to +\infty$.

*Theorem 2.* If there exists a non-empty set of points $B$ of the $(k+1)$ dimensional space of points $(t_0, y_1^{(0)}, \dots, y_k^{(0)})$ which possesses the properties:

(1) $$\inf_B y_s^{(0)} = 0 \qquad (s = 1, \dots k), \quad t_0 \geqslant 0$$

(2) For a certain $\epsilon > 0$ and any $\delta > 0$ there can be found a point $(t_0, y_1^{(0)}, \dots, y_k^{(0)}) \in B$ such that $|Y^{(0)}| < \delta$ and $|Y(t, V^{(0)}, t_0)| < \epsilon$ does not occur for every $t \geqslant t_0$ for all possible choices of continuously differentiable functions $x_1(t), \dots, x_n(t)$, $|X(t)| \leqslant H_2$, where $H_2 < \epsilon$ is a certain positive number, then the null-solution of system (1) is stable.

*Proof.* Suppose the opposite is true. Then for a number $H_2$ according to Definition 1, a number $\delta > 0$ can be found such that

$$|X(t, X^{(0)}, Y^{(0)}, t_0)| < H_2, \qquad |Y(t, X^{(0)}, Y^{(0)}, t_0)| < H_2 \tag{4}$$

for $t \geqslant t_0$ for all $X^{(0)}$ and $Y^{(0)}$ such that $|X^0| < \delta$, $|Y^0| < \delta$. Let us take a point $(t_0, y_1^{(0)}, \dots, y_2^{(0)}) \in B$.

The functions $y_s(t, x_1^{(0)}, \dots, x_n^{(0)}, y^{(0)}, \dots, y_k^{(0)}, t_0)$ may be considered to be the solution of the system (3) in which $x_1(t), \dots, x_n(t)$ are replaced by the functions $(y_1), \dots, y_k^{(0)}, t_0)$ but then, according to condition (2) of the theorem, inequality (3) cannot take place for all $t \geqslant t_0$.

The obtained contradiction shows that the null-solution of the system (1) is unstable. We note a series of special cases of the theorem formulated above.

*Theorem 3.* If:

(1) A null-solution of the system (3) is strongly stable (strongly asymptotically stable),

(2) a null-solution of the system

$$\frac{dx_j}{dt} = g_j(t, x_1, \dots, x_n, 0, \dots, 0) \qquad (j = 1, \dots, n) \tag{5}$$

is uniformly asymptotically stable,

(3) functions $g_j(t_1 x_1, \dots, x_n, y_1, \dots, y_k)$ are continuously differentiable with respect to all their arguments in the domain

$$t \geqslant 0, \quad |X| \leqslant H, \quad |Y| \leqslant H$$

(4) functions

$$\frac{\partial g_j(t, x_1, \dots, x_n, 0, \dots, 0)}{\partial x_i}, \qquad g_j(t, X, Y,) - g_j(t, X, 0)$$

$$\frac{\partial g_j(t, X, Y)}{dt} \qquad \begin{pmatrix} j = 1, \dots, n \\ i = 1, \dots, n \end{pmatrix}$$

are bounded with respect to $t$ uniformly in the domain $|X| < H$, $|Y| < H$, $t \geqslant 0$, then the null-solution of the system (1) also will be stable (asymptotically stable).

*Proof*. For the satisfaction of conditions (2), (3) and (4) of the system (5) there exists a Liapunov function $V(t, x_1, \ldots, x_n)$. It is easy to verify that the function

$$W(t; X, Y) = \frac{\partial V}{\partial t} + \sum_{j=1}^{n} \frac{\partial V}{\partial x_j} g_j$$

is in this case strictly negative-definite; therefore in satisfying conditions (2), (3) and (4), conditions (2) and (3) of Theorem 1 are satisfied, which completes the proof of the present theorem.

*Remarks*. The conditions (2), (3) and (4) of Theorem 3 can be made weaker by using the result obtained in reference [2].

*Theorem 4*. If there exists a certain number $\epsilon > 0$ such that inequality $|V(t, X^{(0)}, Y^{(0)}, t_0| < \epsilon$ is violated for all $t \geqslant t_0 \geqslant 0$, and $Y^{(0)} \neq 0$ is sufficiently small for any choice of the continuously differentiable functions

$$x_j(t) \qquad (j = 1, \ldots, n), \qquad |X(t)| < H_2, \qquad t \geqslant 0, \ H_2 > 0$$

then the null-solution of the system (1) is unstable. Here, as above, $Y(t, Y^{(0)}, t_0)$ denotes the set of functions $y_s$, which represent a solution of the system (2) possessing the property $y_s = y_s^{(0)}$ for $t = t_0$.

*Proof*. Consider a domain $t \geqslant 0$, $|Y| \leqslant \epsilon$. It is easy to see that this domain possesses all properties of domain $B$, formulated in Theorem 2. Then, according to Theorem 2, the null-solution of system (1) is unstable. We note that the first such method of analyzing the problem of the stability of a null-solution of the system of differential equations in doubtful cases was applied by Liapunov, in reference [3].

The same method was developed by I.G. Malkin in reference [4] from which the method of proof of Theorem (1) is taken.

BIBLIOGRAPHY


1. Malkin, I.G., K voprosu ob obrashchenii teoremy Liapunova ob asimptoticheskoi ustoichivosti (On the application of Liapunov's theorem on asymptotic stability). *PMM* Vol. 18, No. 2, 1954.
2. Zubov, V.I., K teorii 2-go metoda Liapunova. (On the theory of Liapunov's second method). *Dokl. Akad. Nauk SSSR* Vol.100, No.5, 1955.

3. Liapunov, A.M., *Obshchaia zadacha ob ustoichivosti dvizheniia (The General Problem of Stability of Motion)*. GITTL, Moscow, Leningrad, 1950.

4. Malkin, I.G., Nekotorye osnovnye teovemy teorii ustoichivosti dvizheniia v kriticheskikh sluchaiakh (Some basic theorems in the theory of stability of motion in critical cases). *PMM* Vol. 6, No. 6, 1942.

*Translated by* M.M.S.